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ANALYSIS OF FORCED VIBRATIONS IN AN INFINITE ICE SHEET

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Report No. 742G-72-10
August, 1972

by

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SUNOCO E & P LIMITED
CALGARY, ALBERTA

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ABSTRACT

A test was conducted to determine the feasibility of evaluating insitu the effective elastic modulus of an ice sheet. A vibratory motion was induced in an ice sheet by heaving a 194 ton air cushion vehicle. The acceleration data was measured at two points on the ice and one point on the vehicle.

A mathematical equation based on the elastic theory of plates was derived to analyze the data. Numerical integration of the equation revealed that the form of the equation results in singularities which prevents the mathematical analysis of the field data. Graphical analysis of the field data showed that the accelrometers on the ice sheet were unable to measure accurately the accelerations of the vibrating ice sheet.

Future efforts will be the development of a mathematical analysis accounting for the viscoelastic characteristics of an ice sheet. This report terminates the effort for an elastic analysis of a vibrating ice sheet.

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I. INTRODUCTION

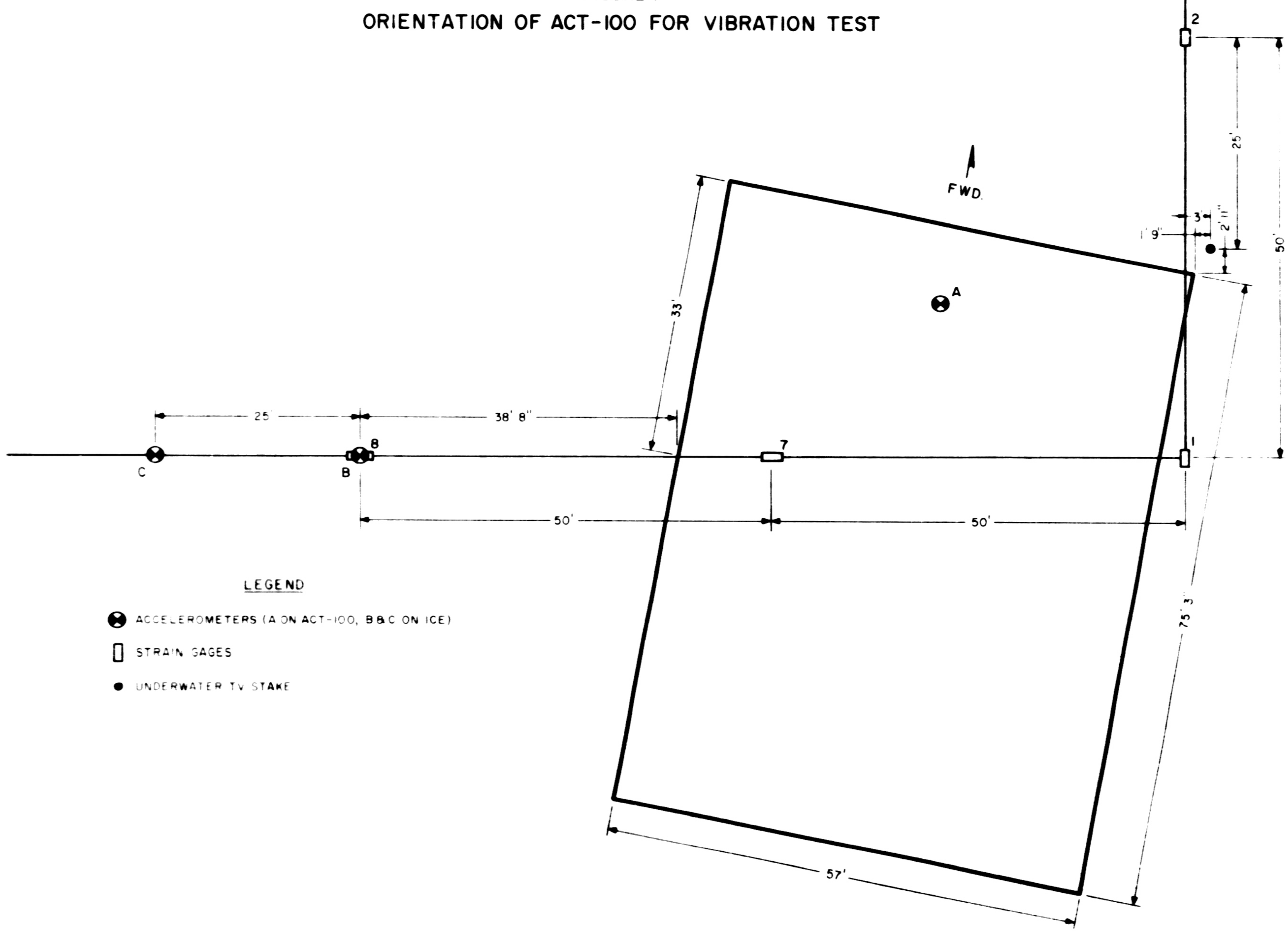
This report describes a theoretical analysis of the vibrating ice sheet problem. Use was made of the ACT-100 vibration data taken at Yellowknife, N.W.T. on February 10, 1972. The goal of this study was to determine the reliability of engineering ice property determinations by vibration techniques.

An attempt was made to simulate mathematically the vibrating motion of a floating ice sheet with a time varying forcing function applied over a finite area. Our mathematical basis was the solution of the equation of motion as obtained by Nevel [1] in 1970. A forcing function for the ice is derived for use in Nevel's solution. A numerical evaluation of the solution was attempted to obtain a time history of ice displacement. Double differentiation of the solution for displacement was to be compared with the acceleration data of the ACT-100 vibration test.

The experimental data of this test was obtained by means of tri-axial accelerometers fixed on the ice surface and on the ACT-100. The output of these accelerometers was reordered on magnetic tape by a Kennedy Incremental Recorder. Refer to Figure 1 for the orientation of the ACT-100 with respect to the accelerometers A, B, and C during the vibration test. All accelerations referred to in this report are those in the vertical direction.

FIGURE I

ORIENTATION OF ACT-100 FOR VIBRATION TEST



II. DERIVATION OF THE FORCING FUNCTION

A forcing function is needed to obtain the general solution of the equation of motion for the ice sheet. This forcing function is unique to the ACT-100 vibration tests.

Since we know the weight of the ACT-100 and its acceleration as a function of time, we can write an expression for the force on the ACT-100 due to the ice. Consider the free body diagram, Figure 2, of a dynamic system composed of the ACT-100 and its air cushion.

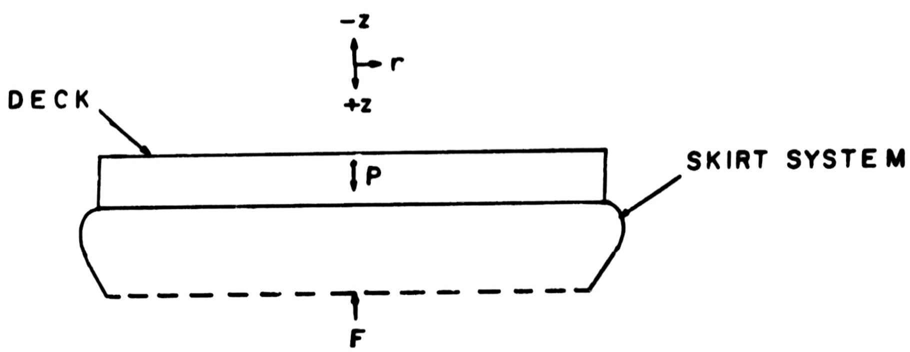


Figure 2

FREE BODY FORCE DIAGRAM OF ACT-100

where

P = weight of the ACT-100

F = force on the ACT-100 due to the ice.

From Newton's second law,

$$\sum \text{Forces} = ma$$

which when applied to our case becomes:

$$P - F = \frac{P}{g} a$$

or $$F = P\left(1 - \frac{a}{g}\right)$$

The force felt by the ice is -F or

$$F_i = P\left(\frac{a}{g} - 1\right). \tag{1}$$

As Figure 3 shows, acceleration data obtained from the vibration test can be simulated by a sinusoidal function for the form,

$$a(t) = A_m \sin \omega t \tag{2}$$

where A_m = amplitude of acceleration

ω = frequency of hovercraft oscillation

t = time

Substituting (2) into (1), the forcing function for the ice is:

$$F_i = P\,f(t) = P\left(\frac{A_m}{g} \sin \omega t - 1\right) \tag{3}$$

Figure 3 presents a superposed plot of the ACT-100 acceleration for several cycles of vibration. These cycles were chosen from one block of test data that seemed to be well behaved (see Table I). This superposition of successive cycles substantiates the validity of the assumptions of steady state motion and sinusoidal nature of the ACT-100 motion.

FIGURE 3-VERTICAL ACCELERATION OF ACT-100

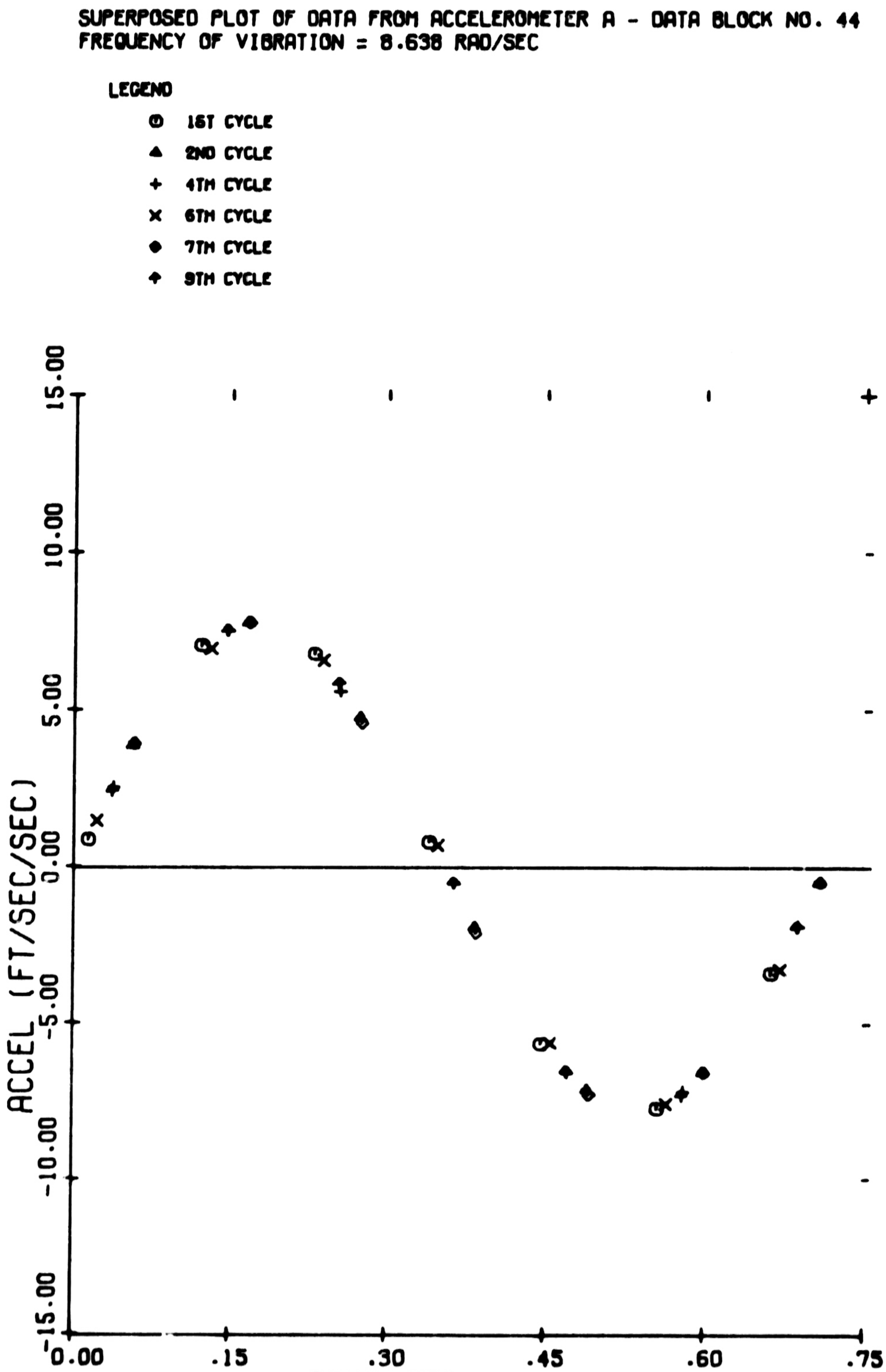


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TABLE I

VERTICAL ACCELERATION DATA*

CYCLE NO.	VERTA	VERTB	VERTC
	3.56	-.03	-.16
	-3.37	.09	-.09
	-7.77	.60	0.00
	-5.88	.60	0.00
1	.88	.13	-.06
	7.05	0.03	-.09
	6.77	-.09	-.13
	.79	.03	-.16
	-5.66	.16	-.09
	-7.74	.60	.03
	-3.40	.60	.03
2	3.87	.03	-.06
	7.77	-.06	-.16
	4.75	-.03	-.16
	-1.92	.03	-.09
	-7.14	.60	-.03
	-6.58	.60	0.00
	-.50	.60	-.03
3	6.20	-.03	-.09
	7.52	-.09	-.16
	-4.56	.13	-.06
	-7.65	.60	0.00
	-4.56	.60	0.00
4	2.55	.06	-.09
	7.52	-.06	-.13
	5.60	-.06	-.13
	-.53	.06	-.09
	-6.58	.60	-.03
	-7.18	.60	.03
	-1.89	.13	0.00
5	5.51	.03	-.09
	7.55	-.06	-.16
	3.49	-.03	-.13
	-3.43	.06	-.06
	-7.58	.60	-.03
	-5.60	.60	0.00
6	1.48	.13	-.03
	6.95	-.03	-.09
	6.58	-.09	-.13
	.69	.03	-.13
	-5.63	.60	-.06
	-7.58	.60	0.00
	-3.27	.60	0.00
7	3.93	.03	-.09
	7.77	-.03	-.16

* The units are ft/sec^2 and the values are taken from data block No. 44. Total load on ice was 244 tons.

III. SOLUTION OF THE GOVERNING EQUATION

As derived by Nevel [1] using the bending theory of thin plates, the differential equation describing the motion of an infinite ice sheet floating on water with an applied forcing function of time, P f(t) is

$$\frac{\partial^2 \bar{w}}{\partial t^2} + \beta^2 \bar{w} = \frac{P f(t)}{\pi \rho g} \frac{\beta^2}{1 + \gamma^4 \ell^4} \frac{J_1 (\gamma R)}{\gamma R} \tag{4}$$

where $\bar{w}$ and γ are the transformed displacement and variable of transformation respectively after the application of a zero order Hankel Transform.

The assumptions made in Nevel's development were:

(1) infinite ice sheet floating on water
(2) ice assumed to be homogeneous, isotropic, and elastic
(3) depth of the water assumed constant
(4) water has no viscosity

The complementary solution of Equation (4) is

$$\bar{W}_c = A \sin \beta t + B \cos \beta t \tag{5}$$

to which must be added the particular solution for the appropriate forcing function.

Putting Equation (4) in operator form and introducing two constants, K_1 and K_2, we have

$$(D^2 + \beta^2) \bar{w} = K_1 \sin \omega t + K_2 \tag{6}$$

Using the method of undetermined coefficients, the particular solution takes the general form:

$$\bar{w} = AK_1 \sin \omega t + BK_1 \cos \omega t + C \tag{7}$$

Differentiating twice with respect to time;

$$D\bar{w} = AK_1 \omega \cos \omega t - BK_1 \omega \sin \omega t \tag{8}$$

$$D^2\bar{w} = -AK_1 \omega^2 \sin \omega t - BK_1 \omega^2 \cos \omega t \tag{9}$$

Substituting Equations (7) and (9) into Equation (6) and rearranging we have;

$$(\beta^2 AK_1 \omega^2)\sin \omega t + (\beta^2 BK_1 - BK_1\ \omega^2)\ \cos \omega t + \beta^2 C = K_1 \sin \omega t + K_2 \quad (10)$$

Evaluating A, B, and C by equating coefficients of like items we obtain;

$$A = \frac{1}{\beta^2 - \omega^2}$$

$$B = 0$$

$$C = \frac{K_2}{\beta^2} \quad (11)$$

Thus the particular solution is

$$\overline{w}_p = \frac{K_1}{\beta^2 - w^2} \sin \omega t + \frac{K_2}{\beta^2} \quad (12)$$

By adding equations (5) and (12), we obtained the total solution

$$\overline{w} = A \sin \beta t + B \cos \beta t + \frac{K_1}{\beta^2 - w^2} \sin \omega t + \frac{K_2}{\beta^2} \quad (13)$$

Using the boundary conditions $w = w_o$ and $\frac{\partial w}{\partial t} = 0$ at time zero, we evaluate the constants A and B;

$$B = w_o - \frac{K_2}{\beta^2}$$

$$A = - \frac{K_1 \omega}{\beta(\beta^2 - \omega^2)}$$

Now taking the inverse Hankel Transform back to the variables r and w, we obtain the equation of displacement as a function of time, t, and distance from the load, r;

$$w(t,r) = \int_0^\infty \gamma \left[- \frac{K_1}{\beta(\beta^2-\omega^2)} \sin \beta t + (w_o - \frac{K_2}{\beta^2}) \cos \beta t + \frac{K_2}{\beta^2} + \frac{K_1}{\beta^2 - \omega^2} \sin \omega t \right] J_o(\gamma r)\, d\gamma \tag{14}$$

The sin βt and cos βt terms go to zero for large time as shown by Titchmarsh [2]. Finally, the equation for the steady state displacement of an infinite sheet of floating ice with the forcing function of Equation (3) as a function of time and radial distance is;

$$w(t,r) = \int_0^\infty \left[\gamma \frac{K_2}{\beta^2} J_o(\gamma r) + \frac{K_1}{\beta^2 - \omega^2} \sin \omega t\, J_o(\gamma r) \right] d\gamma \tag{15}$$

To complete the conversion back into the r and w domain, one must be able to evaluate the integral of Equation (15). This is discussed under the section "Evaluation of the Solution."

IV. EVALUATION OF THE SOLUTION

To complete the transformation of the solution back into the r and w domain, the integral of Equation (15) must be evaluated. An attempt was made to evaluate this integral numerically with the aid of the computer. During this laborius and lengthy effort to apply numerical integration techniques, it was discovered that with certain combinations of values for E and γ, discontinuities of the integrand existed. These singularities of the integrand function render meaningless any conventional, real variable, numerical evaluation of this integral. In light of this information and after consultation with Dr. R. C. Clarke [6], we decided that a further in depth mathematical investigation of the nature and meaning of these singularities is required before conclusions can be reached about the validity and applicability of Nevel's approach to the solution of this problem.

V. GENERAL DISCUSSION

Equation (4) as derived by Nevel [1], assumes the load is applied over a circular area. In our case, the load is applied over a rectangule. An analysis taking into consideration two dimensional stresses in the ice due to a rectangular load is extremely complex and beyond the scope of this study. However, we believe that the use of an equivalent radius for rectangular loads will yield results with sufficient accuracy. This point is a possible source of error which increases as the footprint of a load becomes "less circular" (ratio of length to width increases).

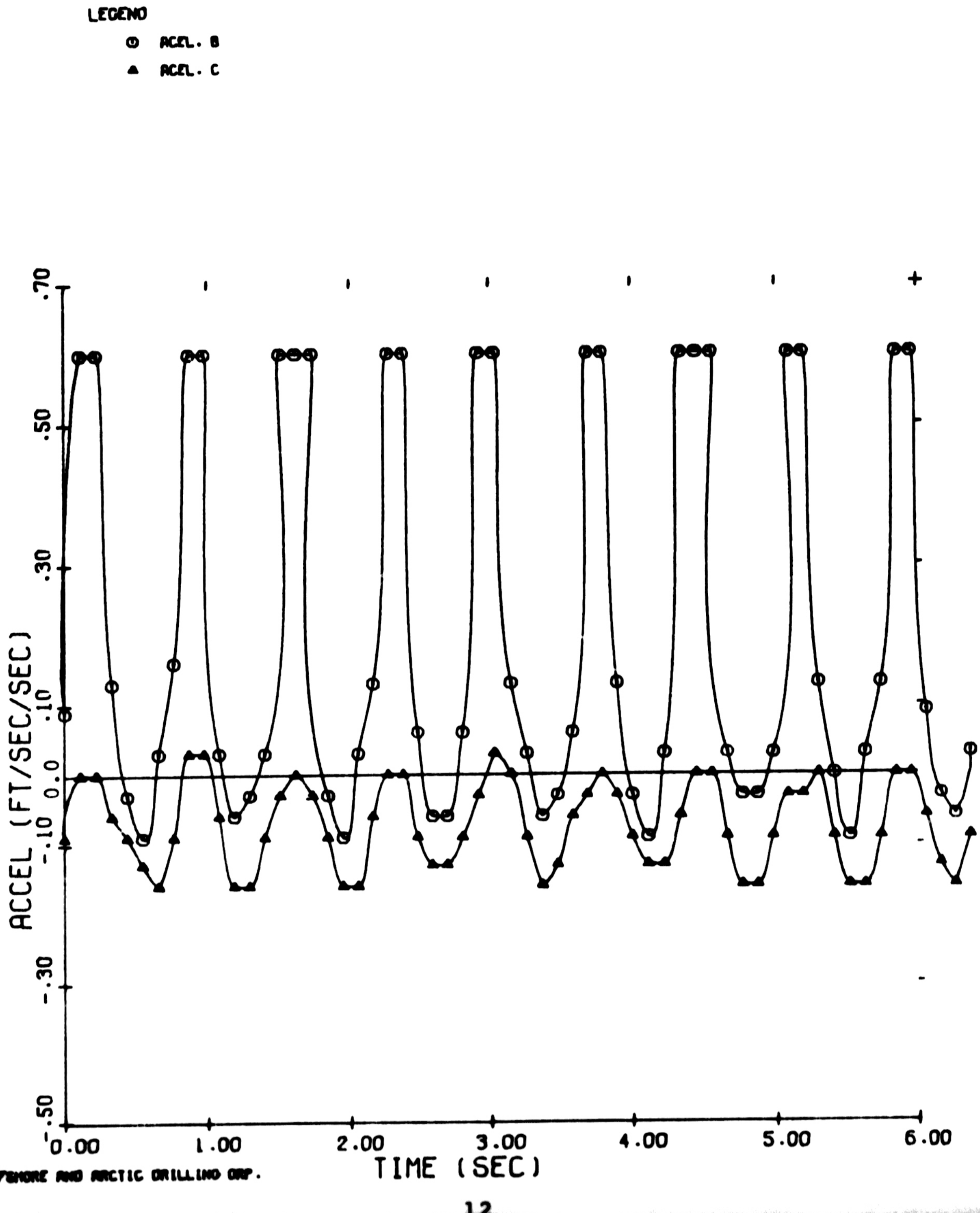
To explore the possibility of evaluating the solution by other means, such as by complex variable integration or by use of asymptotic approximations, would require an estimated man-month. Even then a fruitful result could not be guaranteed. Even if the additional study should prove successful, one must keep in mind Nevel's assumption that the ice behaves as a thin elastic plate. It has been shown [3], [4] and [5], that ice behaves as a visco-elastic material and that elastic theory alone is inadequate to describe the behavior of the ice. Thus, the time that would be spent attempting to evaluate Nevel's elastic solution might be better spent deriving a visco-elastic solution.

If such a solution were developed, comparison to the present field data on ice vibration would be relatively nonconclusive. The most that we can deduce from the present ice vibration data is, for example, that the peak to peak value of ice acceleration was no greater than 0.69 ft/sec^2 at B and 0.19 ft/sec^2 at C. The variation within these peak to peak values was not discernable by the 5g accelerometers. Refer to Figure 4 for a plot of acceleration data from accelerometers B and C from data block No. 44. These accelerations were small relative to the full scale reading (5g) of the accelerometers. This resulted in the data being recorded in increments of magnitude comparible to the amplitude of the acceleration. Compounding this distortion of the acceleration data was the occurrence of a gap in the data from B. It can be seen from Table 1 or Figure 4 that no values of acceleration are recorded from 0.16 ft/sec^2 to 0.6 ft/sec^2, and yet the instrument was capable of detecting approximately 15 increments within this range. The cause of this phenomenon is unknown.

Another feature noticable from Figure 4 is an apparent vertical shift of the data. These curves should be centered about the horizontal axis due to the sinusoidal nature of the ACT-100 acceleration and hence the forcing function and ice motion. This shifting is no doubt another result of the lack of the required accelerometer sensitivity of measurement of accelerations of this magnitude. Although care was taken with the on site calibration of these accelerometers, a small error in calibration has understandably shown up since the amplitude of the phenomenon measured is 1.9% of the full scale reading of the instrument.

FIGURE 4-VERTICAL ACCELERATION OF ICE SHEET

DATA FROM ACCELEROMETERS B AND C - DATA BLOCK NO. 44
ACCELEROMETER B - ON ICE 66 FT. FROM CENTER OF ACT-100
ACCELEROMETER C - ON ICE 91 FT. FROM CENTER OF ACT-100



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VI. CONCLUSIONS

1. The proper mathematical basis has not been developed that will permit us to determine the feasibility of vibration techniques for the measurement of engineering ice properties.

2. We found it impossible to use conventional real variable numerical techniques to evaluate Nevel's solution due to the singularities which exist in the integrand.

3. A sinusoidal forcing function of the type,

$$F_i = P\left(\frac{Am}{g}\sin\omega t - 1\right)$$

was valid for describing the force applied to the ice during the ACT-100 vibration test.

4. The assumption that steady state motion of the ACT-100 had been reached was valid for the data block studied.

VII. RECOMMENDATIONS

1. The effort to advance the elastic solution of this problem to a usable form should be discontinued.

2. Sun should initiate efforts to obtain the visco-elastic solution to the vibrating ice sheet problem.

3. Additional field testing to determine the feasibility of determining the engineering properties of ice by vibration should be resumed only when the proper mathematical basis has been obtained.

VIII. REFERENCES

1. Nevel, D. E., Vibration of a Floating Ice Sheet. Cold Regions Research and Engineering Laboratory, Res. Rpt 281, August, 1970.

2. Titchmarsh, E. C., Theory of Fourier Integrals, 2nd Edition, 1948, pp 11-12.

3. Hill, W. L., Static Load Tests on the Yellowknife Ice Sheet, Report No. 742G-72-7, July, 1972.

4. Frankenstein, G. E., "Strength of Ice Sheets," Proceedings of Conference on Ice Pressures Against Structures, Laval University, Quebec, Canada, 10-11 November, 1966, pp 79-87.

5. Tabata, Tadashi, "Studies on Visco-elastic Properties of Sea Ice," Arctic Sea Ice, Publication 598, National Academy of Sciences - National Research Council, Washington, D. C., December, 1958, pp 139-147.

6. Clarke, R. C. (Dr.) Senior Research Engineer, Geophysical Lithology Studies, personal communications, August, 1972.

IX. NOMENCLATURE

A, B, C	arbitrary constants
a	acceleration, (ft/sec^2)
A_m	amplitude of a sinusoidal acceleration, (ft/sec^2)
D	$Eh^3/[12(1-\mu^2]$ the flexural rigidity
E	modulus of elasticity (lb$_f$/ft^2)
g	acceleration of gravity (ft/sec^2)
h	the thickness of the plate (ft)
K_1	$\frac{P}{\pi \rho g}$ $\frac{\beta^2}{1+\gamma^4 \ell^4}$ $\frac{J_1(\ R)}{\gamma R}$ $\frac{Am}{\varsigma}$
K_2	$\frac{-g}{A_m}$ K_1
k	foundation modulus, $\left(\frac{slugs}{lbm} / ft^3\right)$
m	mass density of the plate, slugs
P	weight of the load, (lb$_f$)
R	equivalent radius of area over which load is applied, (ft)
r & z	cylindrical coordinates, Z positive downward
t	time, (sec)
w	vertical deflection of the plate, (ft)
$\bar{w}$	transformed vertical deflection

w_o static deflection, (ft)

GREEK SYMBOLS

β $\dfrac{g}{\ell}\,\dfrac{1+\gamma^4\ell^4}{\mu+\dfrac{1}{\gamma\ell\tanh\gamma H}}$

γ variable of transformation

μ Poisson's ratio of the plate

ρ mass density of water, slugs

ω frequency of vibration of the load, $\frac{\text{rad}}{\text{sec}}$

SUBSCRIPTS

c complementary solution

i ice

m amplitude

p particular solution

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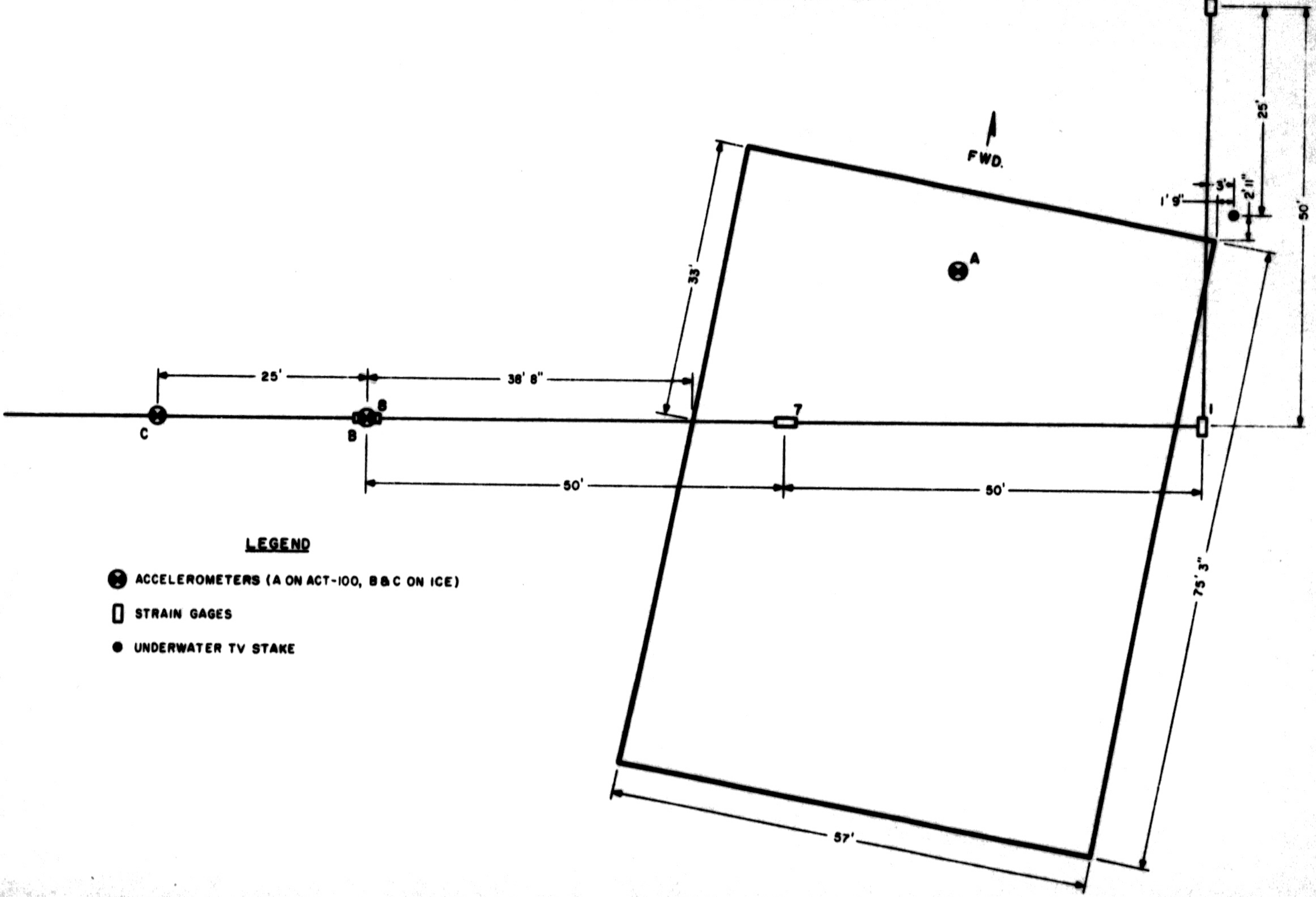
FIGURE I

ORIENTATION OF ACT-100 FOR VIBRATION TEST



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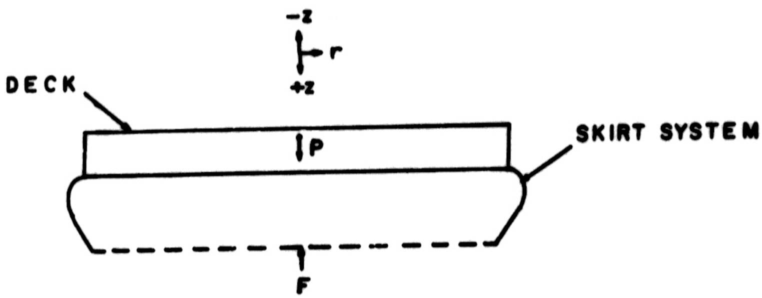

Figure 2

FREE BODY FORCE DIAGRAM OF ACT-100

where

P = weight of the ACT-100

F = force on the ACT-100 due to the ice.

From Newton's second law,

$$\sum \text{Forces} = ma$$

which when applied to our case becomes:

$$P - F = \frac{P}{g} a$$

or $$F = P\left(1 - \frac{a}{g}\right)$$

The force felt by the ice is -F or

$$F_i = P\left(\frac{a}{g} - 1\right). \qquad (1)$$

As Figure 3 shows, acceleration data obtained from the vibration test can be simulated by a sinusoidal function for the form,

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where A_m = amplitude of acceleration

ω = frequency of hovercraft oscillation

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Figure 3 presents a superposed plot of the ACT-100 acceleration for several cycles of vibration. These cycles were chosen from one block of test data that seemed to be well behaved (see Table I). This superposition of successive cycles substantiates the validity of the assumptions of steady state motion and sinusoidal nature of the ACT-100 motion.

FIGURE 3-VERTICAL ACCELERATION OF ACT-100

SUPERPOSED PLOT OF DATA FROM ACCELEROMETER A - DATA BLOCK NO. 44
FREQUENCY OF VIBRATION = 8.638 RAD/SEC

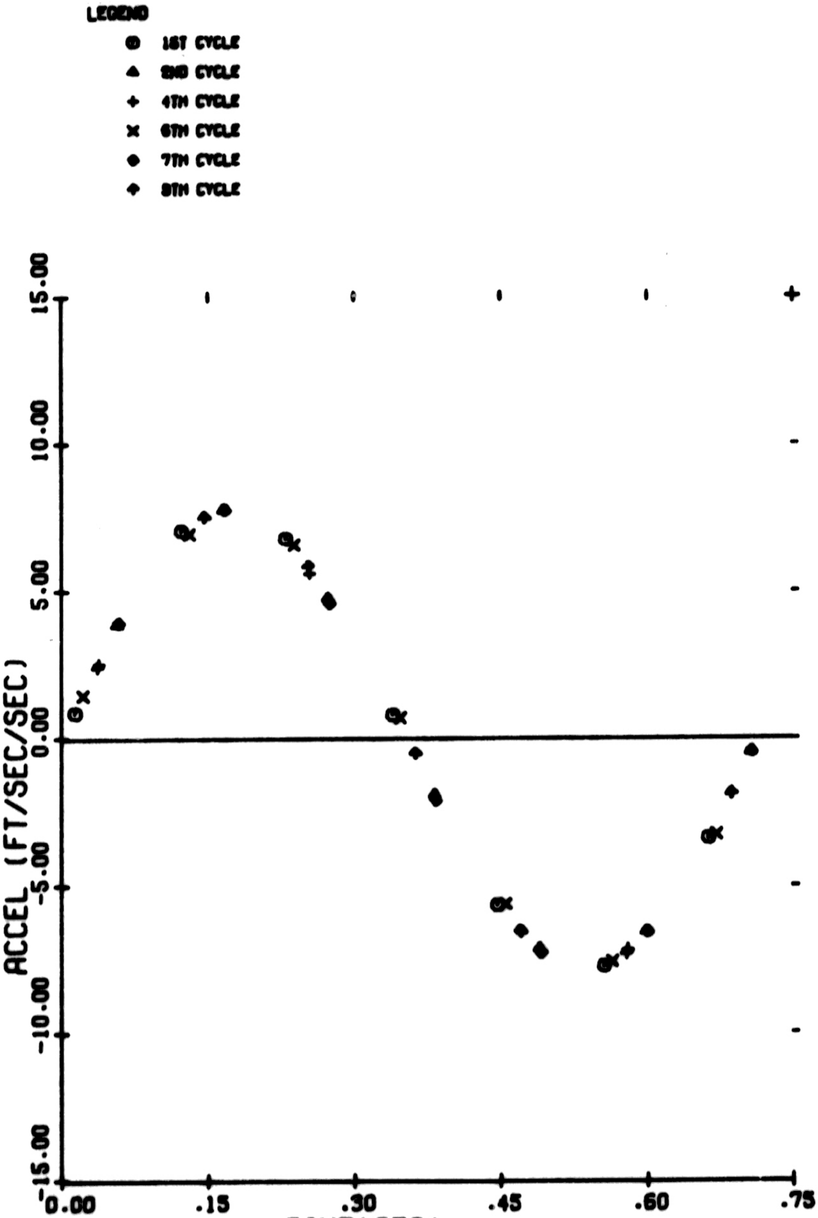

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TABLE I

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	-.53	.06	-.09
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	-3.43	.06	-.06
	-7.58	.60	-.03
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to which must be added the particular solution for the appropriate forcing function.

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$$(D^2 + \beta^2)\, \bar{W} = K_1 \sin \omega t + K_2 \tag{6}$$

Using the method of undetermined coefficients, the particular solution takes the general form:

$$\bar{W} = AK_1 \sin \omega t + BK_1 \cos \omega t + C \tag{7}$$

Differentiating twice with respect to time;

$$D\bar{W} = AK_1\, \omega \cos \omega t - BK_1 \omega \sin \omega t \tag{8}$$

$$D^2\bar{W} = -AK_1\, \omega^2 \sin \omega t - BK_1 \omega^2 \cos \omega t \tag{9}$$

Substituting Equations (7) and (9) into Equation (6) and rearranging we have;

$$(\beta^2 AK_1 \omega^2)\sin \omega t + (\beta^2 BK_1 - BK_1 \omega^2) \cos \omega t + \beta^2 C = K_1 \sin \omega t + K_2 \quad (10)$$

Evaluating A, B, and C by equating coefficients of like items we obtain;

$$A = \frac{1}{\beta^2 - \omega^2}$$

$$B = 0$$

$$C = \frac{K_2}{\beta^2} \quad (11)$$

Thus the particular solution is

$$\bar{w}_p = \frac{K_1}{\beta^2 - w^2} \sin \omega t + \frac{K_2}{\beta^2} \quad (12)$$

By adding equations (5) and (12), we obtained the total solution

$$\bar{w} = A \sin \beta t + B \cos \beta t + \frac{K_1}{\beta^2 - w^2} \sin \omega t + \frac{K_2}{\beta^2} \quad (13)$$

Using the boundary conditions $w = w_o$ and $\frac{\partial w}{\partial t} = 0$ at time zero, we evaluate the constants A and B;

$$B = w_o - \frac{K_2}{\beta^2}$$

$$A = - \frac{K_1 \omega}{\beta(\beta^2 - \omega^2)}$$

Now taking the inverse Hankel Transform back to the variables r and w, we obtain the equation of displacement as a function of time, t, and distance from the load, r;

$$w(t,r) = \int_0^\infty \gamma \left[- \frac{K_1}{\beta(\beta^2 - \omega^2)} \sin \beta t + \left(w_o - \frac{K_2}{\beta^2}\right) \cos \beta t + \frac{K_2}{\beta^2} + \frac{K_1}{\beta^2 - \omega^2} \sin \omega t \right] J_o(\gamma r)\, d\gamma \tag{14}$$

The sin βt and cos βt terms go to zero for large time as shown by Titchmarsh [2]. Finally, the equation for the steady state displacement of an infinite sheet of floating ice with the forcing function of Equation (3) as a function of time and radial distance is;

$$w(t,r) = \int_0^\infty \left[\gamma \frac{K_2}{\beta^2} J_o(\gamma r) + \frac{K_1}{\beta^2 - \omega^2} \sin \omega t\, J_o(\gamma r) \right] d\gamma \tag{15}$$

To complete the conversion back into the r and w domain, one must be able to evaluate the integral of Equation (15). This is discussed under the section "Evaluation of the Solution."

IV. EVALUATION OF THE SOLUTION

To complete the transformation of the solution back into the r and w domain, the integral of Equation (15) must be evaluated. An attempt was made to evaluate this integral numerically with the aid of the computer. During this laborius and lengthy effort to apply numerical integration techniques, it was discovered that with certain combinations of values for E and γ, discontinuities of the integrand existed. These singularities of the integrand function render meaningless any conventional, real variable, numerical evaluation of this integral. In light of this information and after consultation with Dr. R. C. Clarke [6], we decided that a further in depth mathematical investigation of the nature and meaning of these singularities is required before conclusions can be reached about the validity and applicability of Nevel's approach to the solution of this problem.

V. GENERAL DISCUSSION

Equation (4) as derived by Nevel [1], assumes the load is applied over a circular area. In our case, the load is applied over a rectangule. An analysis taking into consideration two dimensional stresses in the ice due to a rectangular load is extremely complex and beyond the scope of this study. However, we believe that the use of an equivalent radius for rectangular loads will yield results with sufficient accuracy. This point is a possible source of error which increases as the footprint of a load becomes "less circular" (ratio of length to width increases).

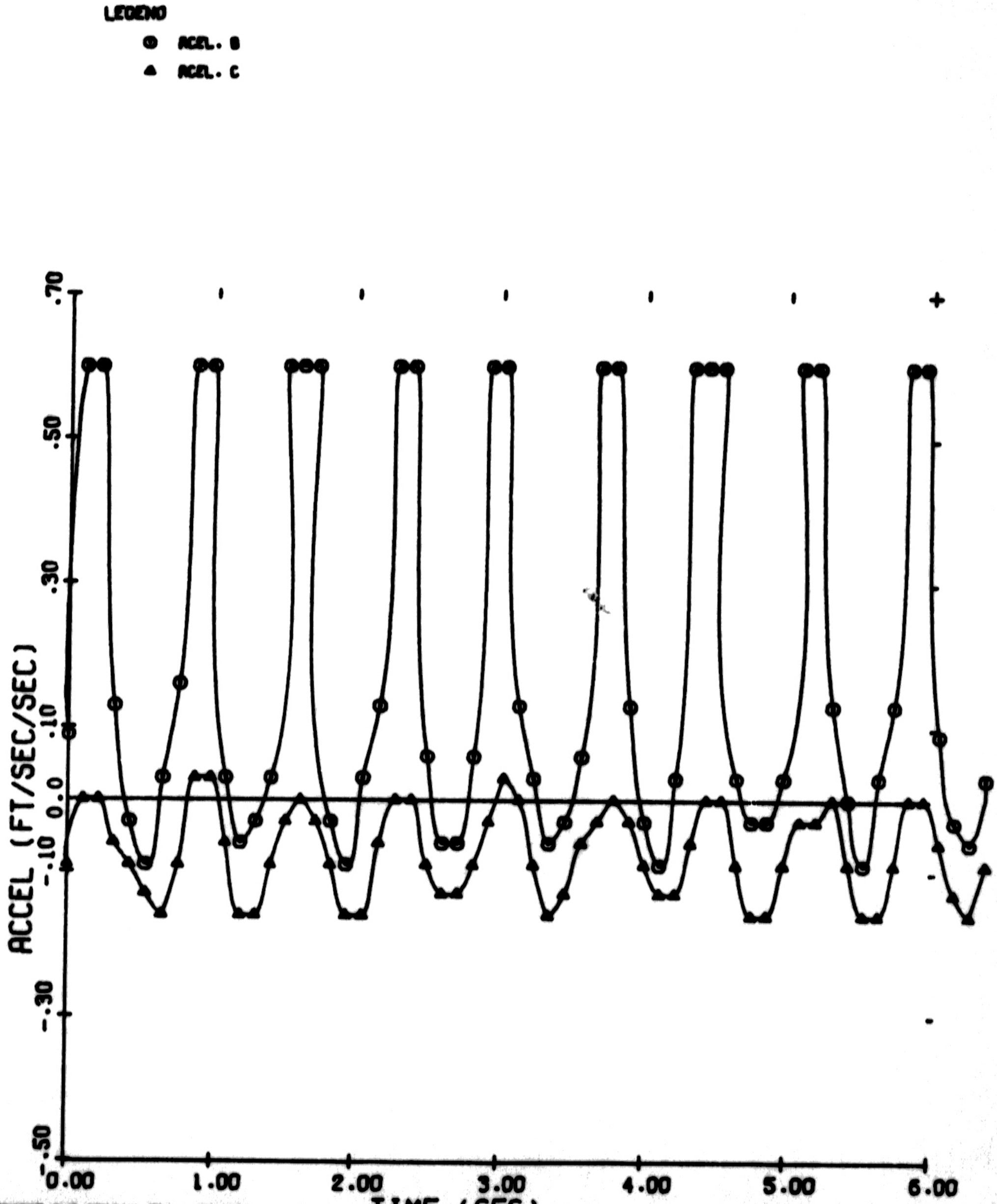
To explore the possibility of evaluating the solution by other means, such as by complex variable integration or by use of asymptotic approximations, would require an estimated man-month. Even then a fruitful result could not be guaranteed. Even if the additional study should prove successful, one must keep in mind Nevel's assumption that the ice behaves as a thin elastic plate. It has been shown [3], [4] and [5], that ice behaves as a visco-elastic material and that elastic theory alone is inadequate to describe the behavior of the ice. Thus, the time that would be spent attempting to evaluate Nevel's elastic solution might be better spent deriving a visco-elastic solution.

If such a solution were developed, comparison to the present field data on ice vibration would be relatively nonconclusive. The most that we can deduce from the present ice vibration data is, for example, that the peak to peak value of ice acceleration was no greater than 0.69 ft/sec^2 at B and 0.19 ft/sec^2 at C. The variation within these peak to peak values was not discernable by the 5g accelerometers. Refer to Figure 4 for a plot of acceleration data from accelerometers B and C from data block No. 44. These accelerations were small relative to the full scale reading (5g) of the accelerometers. This resulted in the data being recorded in increments of magnitude comparible to the amplitude of the acceleration. Compounding this distortion of the acceleration data was the occurrence of a gap in the data from B. It can be seen from Table 1 or Figure 4 that no values of acceleration are recorded from 0.16 ft/sec^2 to 0.6 ft/sec^2, and yet the instrument was capable of detecting approximately 15 increments within this range. The cause of this phenomenon is unknown.

Another feature noticable from Figure 4 is an apparent vertical shift of the data. These curves should be centered about the horizontal axis due to the sinusoidal nature of the ACT-100 acceleration and hence the forcing function and ice motion. This shifting is no doubt another result of the lack of the required accelerometer sensitivity of measurement of accelerations of this magnitude. Although care was taken with the on site calibration of these accelerometers, a small error in calibration has understandably shown up since the amplitude of the phenomenon measured is 1.9% of the full scale reading of the instrument.

FIGURE 4-VERTICAL ACCELERATION OF ICE SHEET

DATA FROM ACCELEROMETERS B AND C - DATA BLOCK NO. 44
ACCELEROMETER B - ON ICE 66 FT. FROM CENTER OF ACT-100
ACCELEROMETER C - ON ICE 91 FT. FROM CENTER OF ACT-100



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VI. CONCLUSIONS

1. The proper mathematical basis has not been developed that will permit us to determine the feasibility of vibration techniques for the measurement of engineering ice properties.

2. We found it impossible to use conventional real variable numerical techniques to evaluate Nevel's solution due to the singularities which exist in the integrand.

3. A sinusoidal forcing function of the type,

$$F_i = P\left(\frac{Am}{g}\sin\omega t - 1\right)$$

was valid for describing the force applied to the ice during the ACT-100 vibration test.

4. The assumption that steady state motion of the ACT-100 had been reached was valid for the data block studied.

VII. RECOMMENDATIONS

1. The effort to advance the elastic solution of this problem to a usable form should be discontinued.

2. Sun should initiate efforts to obtain the visco-elastic solution to the vibrating ice sheet problem.

3. Additional field testing to determine the feasibility of determining the engineering properties of ice by vibration should be resumed only when the proper mathematical basis has been obtained.

VIII. REFERENCES

1. Nevel, D. E., Vibration of a Floating Ice Sheet. Cold Regions Research and Engineering Laboratory, Res. Rpt 281, August, 1970.

2. Titchmarsh, E. C., Theory of Fourier Integrals, 2nd Edition, 1948, pp 11-12.

3. Hill, W. L., Static Load Tests on the Yellowknife Ice Sheet, Report No. 742G-72-7, July, 1972.

4. Frankenstein, G. E., "Strength of Ice Sheets," Proceedings of Conference on Ice Pressures Against Structures, Laval University, Quebec, Canada, 10-11 November, 1966, pp 79-87.

5. Tabata, Tadashi, "Studies on Visco-elastic Properties of Sea Ice," Arctic Sea Ice, Publication 598, National Academy of Sciences - National Research Council, Washington, D. C., December, 1958, pp 139-147.

6. Clarke, R. C. (Dr.) Senior Research Engineer, Geophysical Lithology Studies, personal communications, August, 1972.

IX. NOMENCLATURE

A, B, C	arbitrary constants
a	acceleration, (ft/sec^2)
A_m	amplitude of a sinusoidal acceleration, (ft/sec^2)
D	$Eh^3/[12(1-\mu^2]$ the flexural rigidity
E	modulus of elasticity (lb_f/ft^2)
g	acceleration of gravity (ft/sec^2)
h	the thickness of the plate (ft)
K_1	$\frac{P}{\pi \rho g}$ $\frac{\beta^2}{1+\gamma^4 \ell^4}$ $\frac{J_1(R)}{\gamma R}$ $\frac{Am}{\varsigma}$
K_2	$\frac{-g}{A_m}$ K_1
k	foundation modulus, $\left(\frac{slugs}{lbm} / ft^3\right)$
m	mass density of the plate, slugs
P	weight of the load, (lb_f)
R	equivalent radius of area over which load is applied, (ft)
r & z	cylindrical coordinates, Z positive downward
t	time, (sec)
w	vertical deflection of the plate, (ft)
$\bar{W}$	transformed vertical deflection

w_o static deflection, (ft)

GREEK SYMBOLS

β $\frac{q}{\ell} \frac{1 + \gamma^4 \ell^4}{\mu + \frac{1}{\gamma \ell \tanh \gamma H}}$

γ variable of transformation

μ Poisson's ratio of the plate

ρ mass density of water, slugs

ω frequency of vibration of the load, $\frac{\text{rad}}{\text{sec}}$

SUBSCRIPTS

c complementary solution

i ice

m amplitude

p particular solution